

REGULATORS FOR RANKIN-SELBERG PRODUCTS OF MODULAR FORMS

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ABSTRACT. We prove a weak version of Beilinson's conjecture for non-critical values of L -functions for the Rankin-Selberg product of two modular forms.

INTRODUCTION

In his fundamental paper [2, §6], Beilinson introduced the so-called *Beilinson-Flach elements* in the higher Chow group of a product of two modular curves and related their image under the regulator map to special values of Rankin L -series of the form $L(f \otimes g, 2)$, where f and g are newforms of weight 2, as predicted by his conjectures on special values of L -functions. These elements were later exploited by Flach [11] to prove the finiteness of the Selmer group associated to the symmetric square of an elliptic curve. More recently, Bertolini, Darmon and Rotger [4] established a p -adic analogue of Beilinson's result, while Lei, Loeffler and Zerbes [18] constructed a cyclotomic Euler system whose bottom layer are the Beilinson-Flach elements. These results have many important arithmetic applications ([5], [18]).

Our aim in this paper is to define an analogue of the Beilinson-Flach elements in the motivic cohomology of a product of two Kuga-Sato varieties and to prove an analogue of Beilinson's formula for special values of Rankin L -series associated to newforms f and g of any weight ≥ 2 . More precisely, we prove the following theorem.

Theorem 0.1. *Let $f \in S_{k+2}(\Gamma_1(N_f), \chi_f)$ and $g \in S_{\ell+2}(\Gamma_1(N_g), \chi_g)$ be newforms with $k, \ell \geq 0$. Assume that the Dirichlet character χ modulo $N = \text{lcm}(N_f, N_g)$ induced by $\chi_f \chi_g$ is non-trivial. Let j be an integer satisfying $0 \leq j \leq \min\{k, \ell\}$. Assume that the automorphic factor $R_{f,g,N}(j+1)$ defined in Section 5 is non-zero (this holds for example if $\gcd(N_f, N_g) = 1$ or if $k + \ell - 2j \notin \{0, 1, 2\}$). Then the weak version of Beilinson's conjecture for $L(f \otimes g, k + \ell + 2 - j)$ holds.*

The range of critical values (in the sense of Deligne) for the Rankin-Selberg L -function $L(f \otimes g, s)$ is given by $\min\{k, \ell\} + 2 \leq s \leq \max\{k, \ell\} + 1$, so that our L -value $L(f \otimes g, k + \ell + 2 - j)$ is *non-critical*. In fact, the integers $0 \leq j \leq \min\{k, \ell\}$ are precisely those at which the dual L -function $L(f^* \otimes g^*, s + 1)$ vanishes at order 1.

We refer to Theorem 6.4 for the explicit formula giving the regulator of our generalized Beilinson-Flach elements. In the weight 2 case, an explicit version of Beilinson's formula for $L(f \otimes g, 2)$, similar to Theorem 6.4, was proved by Baba and Sreekantan [1] and by Bertolini, Darmon and Rotger [4]. In the higher weight case, a similar formula for the regulator of generalized Beilinson-Flach elements was proved by Scholl (unpublished) and recently by Kings, Loeffler and Zerbes [17]. As a difference with [17], we work directly with the motivic cohomology of the Kuga-Sato varieties (instead of motivic cohomology with coefficients), and we prove that our generalized Beilinson-Flach elements extend to the boundary of the Kuga-Sato varieties (see Sections 7 and 8). Another interesting problem is the integrality of the generalized Beilinson-Flach elements. In the case f and g have weight 2, Scholl proved that if g is not a twist of f , then the Beilinson-Flach elements belong to the integral subspace of motivic cohomology [22, Theorem 2.3.4]. We do not investigate integrality in this article, but it would be interesting to do so using Scholl's techniques.

The plan of this article is as follows. In Section 1, we recall the statement of Beilinson's conjecture for Grothendieck motives. In Section 2, we recall some basic results about motives of modular forms and describe explicitly the Deligne cohomology group associated to the Rankin product of two modular forms. After recalling Beilinson's theory of the Eisenstein symbol (Section 3), we construct in Section 4 special elements $\Xi^{k,\ell,j}(\beta)$ in the motivic cohomology of the product of two Kuga-Sato varieties. After recalling standard facts about the Rankin-Selberg L -function (Section 5), we compute the regulator of our elements $\Xi^{k,\ell,j}(\beta)$

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in Section 6. We then show in Sections 7 and 8, using motivic techniques, that a suitable modification of the elements $\Xi^{k,\ell,j}(\beta)$ extends to the boundary of the Kuga-Sato varieties. Finally, we give in Section 9 the application of our results to Beilinson's conjecture.

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1. BEILINSON'S CONJECTURE

Let X be a smooth projective variety over \mathbb{Q} . For a non-negative integer i and an integer j , let $H_{\mathcal{M}}^i(X, \mathbb{Q}(j))$ be the motivic cohomology and $H_{\mathcal{D}}^i(X_{\mathbb{R}}, \mathbb{R}(j))$ be the Deligne cohomology. Then one can define natural \mathbb{Q} -structures $\mathcal{B}_{i,j}$ and $\mathcal{D}_{i,j}$ in $\det_{\mathbb{R}}(H_{\mathcal{D}}^{i+1}(X_{\mathbb{R}}, \mathbb{R}(j)))$ (see Deninger-Scholl [10, 2.3.2] or Nekovář [19, (2.2)]). Denote the integral part of motivic cohomology by $H_{\mathcal{M}}^i(X, \mathbb{Q}(j))_{\mathbb{Z}}$. Then Beilinson defined the regulator map

$$r_{\mathcal{D}} : H_{\mathcal{M}}^{i+1}(X, \mathbb{Q}(j)) \rightarrow H_{\mathcal{D}}^{i+1}(X_{\mathbb{R}}, \mathbb{R}(j))$$

and formulated a conjecture for the special values of the L -function $L(h^i(X), s)$ as follows.

Conjecture 1.1 (Beilinson [2]). *Assume $j > (i + 2)/2$.*

- (1) *The map $r_{\mathcal{D}} \otimes \mathbb{R} : H_{\mathcal{M}}^{i+1}(X, \mathbb{Q}(j))_{\mathbb{Z}} \otimes \mathbb{R} \rightarrow H_{\mathcal{D}}^{i+1}(X_{\mathbb{R}}, \mathbb{R}(j))$ is an isomorphism.*
- (2) *We have $r_{\mathcal{D}}(\det H_{\mathcal{M}}^{i+1}(X, \mathbb{Q}(j))_{\mathbb{Z}}) = L(h^i(X), j) \cdot \mathcal{D}_{i,j} = L^*(h^i(X), i + 1 - j) \cdot \mathcal{B}_{i,j}$, where $L^*(h^i(X), m)$ is the leading term of the Taylor expansion of $L(h^i(X), s)$ at $s = m$.*

In the case of the near-central point $j = (i + 2)/2$, we have the following modified conjecture. Let $N^{j-1}(X) = \text{CH}^{j-1}(X)_{\text{hom}} \otimes \mathbb{Q}$ be the group of $(j - 1)$ -codimensional cycles modulo homological equivalence. The cycle class map into de Rham cohomology defines an extended regulator map

$$\hat{r}_{\mathcal{D}} : H_{\mathcal{M}}^{i+1}(X, \mathbb{Q}(j)) \oplus N^{j-1}(X) \rightarrow H_{\mathcal{D}}^{i+1}(X_{\mathbb{R}}, \mathbb{R}(j)).$$

Conjecture 1.2 (Beilinson [2]). *Assume $j = (i + 2)/2$.*

- (1) *(Tate's conjecture) We have $\text{ord}_{s=j} L(h^i(X), s) = -\dim_{\mathbb{Q}} N^{j-1}(X)$.*
- (2) *The map $\hat{r}_{\mathcal{D}} \otimes \mathbb{R} : (H_{\mathcal{M}}^{i+1}(X, \mathbb{Q}(j))_{\mathbb{Z}} \oplus N^{j-1}(X)) \otimes \mathbb{R} \rightarrow H_{\mathcal{D}}^{i+1}(X_{\mathbb{R}}, \mathbb{R}(j))$ is an isomorphism.*
- (3) *We have $\hat{r}_{\mathcal{D}}(\det(H_{\mathcal{M}}^{i+1}(X, \mathbb{Q}(j))_{\mathbb{Z}} \oplus N^{j-1}(X))) = L^*(h^i(X), j) \cdot \mathcal{D}_{i,j} = L^*(h^i(X), j - 1) \cdot \mathcal{B}_{i,j}$.*

We will now formulate a version of Beilinson's conjectures for Grothendieck motives. Let $M = (X, p)$ be a Grothendieck motive over \mathbb{Q} with coefficients in L , where X is a smooth projective variety over \mathbb{Q} and p is a projector in $\text{CH}^{\dim X}(X \times X)_{\text{hom}} \otimes_{\mathbb{Q}} L$. We define the Deligne cohomology of M by

$$H_{\mathcal{D}}(M, j) = p_*(H_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(j)) \otimes L).$$

Let us assume that M is a direct factor of $h^i(X) \otimes L$. We have an L -function $L(M, s) = L(H^i(M), s)$ taking values in $L \otimes \mathbb{C}$. Moreover, there are natural L -structures $\mathcal{B}_{i,j}(M)$ and $\mathcal{D}_{i,j}(M)$ in $\det_{L \otimes \mathbb{R}} H_{\mathcal{D}}^{i+1}(M, j)$. We define Beilinson's regulator as

$$r_{\mathcal{D}} : H_{\mathcal{M}}^{i+1}(X, \mathbb{Q}(j)) \otimes L \rightarrow H_{\mathcal{D}}^{i+1}(X_{\mathbb{R}}, \mathbb{R}(j)) \otimes L \rightarrow H_{\mathcal{D}}^{i+1}(M, j),$$

where the last map is the projection induced by p_* . Similarly, we define an extended regulator map $\hat{r}_{\mathcal{D}}$ in the case $j = (i + 2)/2$. Assume Conjecture 1.1 (1) for X . Then Beilinson's conjecture for $L(M, s)$ can be formulated as follows.

Conjecture 1.3. (1) *If $j > (i + 2)/2$, then $p_*(r_{\mathcal{D}}(\det_L H_{\mathcal{M}}^{i+1}(X, \mathbb{Q}(j))_{\mathbb{Z}} \otimes L)) = L(M, j) \cdot \mathcal{D}_{i,j}(M)$.*
 (2) *If $j = (i + 2)/2$, then $p_*(\hat{r}_{\mathcal{D}}(\det_L(H_{\mathcal{M}}^{i+1}(X, \mathbb{Q}(j))_{\mathbb{Z}} \oplus N^{j-1}(X)) \otimes L)) = L^*(M, j) \cdot \mathcal{D}_{i,j}(M)$.*

Since Conjecture 1.1(1) is in general out of reach, we formulate a weak version of Conjecture 1.3 as follows.

Conjecture 1.4 (Weak version). *Assume M is a direct factor of $h^i(X) \otimes L$. Let $M^{\vee} = (X, {}^t p, i)$ be the dual motive of M .*

- (1) *If $j > (i + 2)/2$, then there exists a subspace V of $H_{\mathcal{M}}^{i+1}(X, \mathbb{Q}(j))$ such that $p_*(r_{\mathcal{D}}(V \otimes L))$ is an L -structure of $H_{\mathcal{D}}^{i+1}(M, j)$ and*

$$\det p_*(r_{\mathcal{D}}(V \otimes L)) = L(M, j) \cdot \mathcal{D}_{i,j}(M) = L^*(M^{\vee}, 1 - j) \cdot \mathcal{B}_{i,j}(M).$$

- (2) If $j = (i+2)/2$, then there exists a subspace V of $H_{\mathcal{M}}^{i+1}(X, \mathbb{Q}(j)) \oplus N^{j-1}(X)$ such that $p_*(\hat{r}_{\mathcal{D}}(V \otimes L))$ is an L -structure of $H_{\mathcal{D}}^{i+1}(M, j)$ and

$$\det p_*(\hat{r}_{\mathcal{D}}(V \otimes L)) = L^*(M, j) \cdot \mathcal{D}_{i,j}(M) = L^*(M^\vee, 1-j) \cdot \mathcal{B}_{i,j}(M).$$

Remark 1.5. We could have required a stronger property in Conjecture 1.4, namely that V is a subspace of $H_{\mathcal{M}}^{i+1}(X, \mathbb{Q}(j))_{\mathbb{Z}}$. But since we don't consider the problem of integrality of elements of motivic cohomology in this paper, we leave Conjecture 1.4 as it is.

2. MOTIVES ASSOCIATED TO MODULAR FORMS

Let us recall some basic properties of motives associated to modular forms. Let $Y = Y(N)$ be the modular curve with full level N -structure defined over \mathbb{Q} and $j : Y \hookrightarrow X = X(N)$ the smooth compactification. Let $\pi : E \rightarrow Y$ be the universal elliptic curve over Y and $\bar{\pi} : \bar{E} \rightarrow X$ be the universal generalized elliptic curve. Then \bar{E} is smooth and proper over \mathbb{Q} . For a non-negative integer k , denote the k -fold fiber product of E over Y by E^k and the k -fold fiber product of \bar{E} over X by \bar{E}^k . Let \hat{E}^k be the Néron model of E^k over X and $\hat{E}^{k,*}$ the connected component. If $k \geq 2$, then \bar{E}^k is singular. Let $\bar{\bar{E}}^k \rightarrow \bar{E}^k$ be the canonical desingularization constructed by Deligne.

Let $f \in S_{k+2}(\Gamma_1(N), \chi)^{\text{new}}$ ($k \geq 0$) be a normalized eigenform. Let $K_f \subset \mathbb{C}$ be the number field generated by the Fourier coefficients of f . Let $M(f)$ be the Grothendieck motive associated to f [21]. It is a motive of rank 2 defined over \mathbb{Q} with coefficients in K_f . The motive $M(f)$ is a direct factor of $h^{k+1}(\bar{\bar{E}}^k) \otimes K_f$. By Grothendieck's theorem, we have an isomorphism $H_B^{k+1}(M(f)) \otimes \mathbb{C} \cong H_{\text{dR}}^{k+1}(M(f)) \otimes \mathbb{C}$ between Betti and de Rham cohomology. The $K_f \otimes \mathbb{C}$ -module $H_{\text{dR}}^{k+1}(M(f)) \otimes \mathbb{C}$ is free of rank 2, with basis $\{\omega_f, \overline{\omega_f}\}$, where

$$\omega_f = (2\pi i)^{k+1} f(\tau) d\tau \wedge dz_1 \wedge \cdots \wedge dz_k.$$

We denote $\omega'_f = G(\chi)^{-1} \omega_f$, where

$$G(\chi) = \sum_{u=1}^{N_\chi} \chi(u) e^{2\pi i u / N_\chi}$$

is the Gauss sum of the Dirichlet character χ and N_χ is the conductor of χ . By [17, Lemma 6.1.1], we have

$$\text{Fil}^i H_{\text{dR}}^{k+1}(M(f)) = \begin{cases} H_{\text{dR}}^{k+1}(M(f)) & \text{if } i \leq 0, \\ K_f \cdot \omega'_f & \text{if } 1 \leq i \leq k+1, \\ 0 & \text{if } i \geq k+2. \end{cases}$$

By Poincaré duality, we have a perfect pairing of K_f -vector spaces

$$H_B^{k+1}(M(f^*)(k+1)) \times H_B^{k+1}(M(f)) \rightarrow K_f.$$

Now, let $f \in S_{k+2}(\Gamma_1(N_f), \chi_f)^{\text{new}}$, $g \in S_{\ell+2}(\Gamma_1(N_g), \chi_g)^{\text{new}}$ ($\ell \geq k \geq 0$) be normalized eigenforms. We consider the Grothendieck motive

$$M(f \otimes g) := M(f) \otimes M(g).$$

This motive has coefficients in $K_{f,g} := K_f K_g$ and is a direct factor of

$$h^{k+1}(\bar{\bar{E}}_{N_f}^k) \otimes h^{\ell+1}(\bar{\bar{E}}_{N_g}^\ell) \otimes K_{f,g} \subset h^{k+\ell+2}(\bar{\bar{E}}_{N_f}^k \times \bar{\bar{E}}_{N_g}^\ell) \otimes K_{f,g}.$$

Let j be an integer such that $0 \leq j \leq k$ and put $n = k + \ell + 2 - j$. The Deligne cohomology of $M(f \otimes g)(n)$ can be expressed as follows. The de Rham realization

$$H_{\text{dR}}^{k+\ell+2}(M(f \otimes g)) = H_{\text{dR}}^{k+1}(M(f)) \otimes H_{\text{dR}}^{\ell+1}(M(g))$$

has dimension 4 over $K_{f,g}$. Moreover $\text{Fil}^n H_{\text{dR}}^{k+\ell+2}(M(f \otimes g))$ is the $K_{f,g}$ -line generated by $\omega'_f \otimes \omega'_g$. Then we have an exact sequence

$$(2.1) \quad 0 \rightarrow \text{Fil}^n H_{\text{dR}}^{k+\ell+2}(M(f \otimes g)) \otimes \mathbb{R} \rightarrow H_B^{k+\ell+2}(M(f \otimes g)(n-1))^+ \otimes \mathbb{R} \rightarrow H_{\mathcal{D}}^{k+\ell+3}(M(f \otimes g)(n)) \rightarrow 0.$$

In particular $H_{\mathcal{D}}^{k+\ell+3}(M(f \otimes g)(n))$ is a free $K_{f,g} \otimes \mathbb{R}$ -module of rank 1.

The exact sequence (2.1) induces a $K_{f,g}$ -rational structure on $H_{\mathcal{D}}^{k+\ell+3}(M(f \otimes g)(n))$. Let us make explicit a generator of this rational structure. Let e_f^\pm be a K_f -basis of $H_B^{k+1}(M(f))^\pm$, and let e_g^\pm be a K_g -basis

of $H_B^{\ell+1}(M(g))^{\pm}$. Under the comparison isomorphism $H_B^{k+1}(M(f)) \otimes \mathbb{C} \cong H_{\text{dR}}^{k+1}(M(f)) \otimes \mathbb{C}$, we have $\omega_f = \alpha_f^+ e_f^+ + \alpha_f^- e_f^-$ for some $\alpha_f^+, \alpha_f^- \in \mathbb{C}$. Note that $\alpha_f^+ \in \mathbb{R}$ and $\alpha_f^- \in i\mathbb{R}$. Similarly, let $\omega_g = \alpha_g^+ e_g^+ + \alpha_g^- e_g^-$. The $K_{f,g}$ -vector space $H_B^{k+\ell+2}(M(f \otimes g)(n-1))^+$ admits as a $K_{f,g}$ -basis (e_1, e_2) where

$$\begin{aligned} e_1 &= e_f^+ \otimes e_g^{(-1)^{n+1}} \otimes (2\pi i)^{n-1}, \\ e_2 &= e_f^- \otimes e_g^{(-1)^n} \otimes (2\pi i)^{n-1}. \end{aligned}$$

The image of $\omega'_f \otimes \omega'_g$ in $H_B^{k+\ell+2}(M(f \otimes g)(n-1))^+ \otimes \mathbb{R}$ under (2.1) is given by

$$\begin{aligned} \pi(\omega'_f \otimes \omega'_g) &= G(\chi_f)^{-1} G(\chi_g)^{-1} \alpha_f^- \alpha_g^{(-1)^n} e_f^- \otimes e_g^{(-1)^n} + G(\chi_f)^{-1} G(\chi_g)^{-1} \alpha_f^+ \alpha_g^{(-1)^{n+1}} e_f^+ \otimes e_g^{(-1)^{n+1}} \\ &= G(\chi_f)^{-1} G(\chi_g)^{-1} (2\pi i)^{1-n} (\alpha_f^+ \alpha_g^{(-1)^{n+1}} e_1 + \alpha_f^- \alpha_g^{(-1)^n} e_2). \end{aligned}$$

Thus a rational structure of $H_D^{k+\ell+3}(M(f \otimes g)(n))$ is given by

$$t := G(\chi_f) G(\chi_g) (2\pi i)^{n-1} (\alpha_f^- \alpha_g^{(-1)^n})^{-1} e_1.$$

Since $M(f \otimes g)(n-1)^\vee \cong M(f^* \otimes g^*)(j+1)$, we have a perfect pairing

$$H_B^{k+\ell+2}(M(f \otimes g)(n-1)) \times H_B^{k+\ell+2}(M(f^* \otimes g^*)(j+1)) \rightarrow K_{f,g}.$$

Now, let us define a canonical element $\Omega \in H_B^{k+\ell+2}(M(f^* \otimes g^*)(j+1)) \otimes \mathbb{C}$, which we will use to pair with the regulator of our generalized Beilinson-Flach element. Under the canonical isomorphism

$$\phi_{\text{dR}} : H_{\text{dR}}^{k+\ell+2}(M(f^*) \otimes M(g^*)(j+1)) \xrightarrow{\cong} H_{\text{dR}}^{k+\ell+2}(M(f) \otimes M(\chi_f) \otimes M(g) \otimes M(\chi_g)(j+1)),$$

the element $G(\overline{\chi_f})^{-1} G(\overline{\chi_g})^{-1} \omega_{f^*} \otimes \omega_{g^*}$ corresponds to a $K_{f,g}^\times$ -rational multiple of $\omega'_f \otimes \omega(\chi_f) \otimes \omega'_g \otimes \omega(\chi_g)$, where $\omega(\chi_f)$ is basis of $H_{\text{dR}}^0(M(\chi_f))$.

We recall the periods for motives associated to Dirichlet characters with coefficients in E . Let $M(\chi)$ be the motive associated to χ with coefficients in a number field E . Then the period of the comparison isomorphism $H_B^0(M(\chi)) = E(\chi) \rightarrow H_{\text{dR}}^0(M(\chi)) = G(\chi) \cdot E$ is given by $G(\chi)^{-1}$, where $E(\chi)$ is the rank one E -vector space on which the Galois group $\text{Gal}(\mathbb{Q}(e^{2\pi i/N_\chi})/\mathbb{Q})$ acts via χ and $G(\chi) \cdot E$ is the E -vector space generated by $G(\chi)$ (for details, see [7, Section 6]).

Under the comparison isomorphism

$$\phi : H_B^{k+\ell+2}(M(f) \otimes M(\chi_f) \otimes M(g) \otimes M(\chi_g)(j+1)) \otimes \mathbb{C} \xrightarrow{\cong} H_{\text{dR}}^{k+\ell+2}(M(f) \otimes M(\chi_f) \otimes M(g) \otimes M(\chi_g)(j+1)) \otimes \mathbb{C},$$

we have

$$\phi^{-1}(\omega'_f \otimes \omega(\chi_f) \otimes \omega'_g \otimes \omega(\chi_g)) = (\alpha_f^+ e_f^+ + \alpha_f^- e_f^-) \otimes (\alpha_g^+ e_g^+ + \alpha_g^- e_g^-) \otimes e(\chi_f) \otimes e(\chi_g).$$

Let $e_f^{\pm, \vee}$ be a K_f -basis of $H_B^{k+1}(M(f)^\vee)^{\pm}$ with $\langle e_f^{\pm, \vee}, e_f^{\pm, \vee} \rangle = 1$, and let $e_g^{\pm, \vee}$ be a K_g -basis of $H_B^{\ell+1}(M(g)^\vee)^{\pm}$ with $\langle e_g^{\pm, \vee}, e_g^{\pm, \vee} \rangle = 1$. We have an isomorphism $\phi_B(f) : H_B^{k+1}(M(f) \otimes M(\chi_f)) \xrightarrow{\cong} H_B^{k+1}(M(f)^\vee(-k-1))$ sending $e_f^{\pm} \otimes e(\chi_f)$ to a $K_{f,g}^\times$ -rational multiple of $(2\pi i)^{-k-1} e_f^{\mp, \vee}$. Note that $(2\pi i)^{k+1} e(\chi_f) \in H_B^0(M(\chi_f)(k+1))^-$, since $\chi_f(-1) = (-1)^k$. Therefore we have an isomorphism

$$\phi_B : H_B^{k+\ell+2}(M(f) \otimes M(g) \otimes M(\chi_f) \otimes M(\chi_g)(j+1)) \xrightarrow{\cong} H_B^{k+\ell+2}(M(f)^\vee \otimes M(g)^\vee(1-n))$$

sending $(2\pi i)^{j+1} e_f^{\pm} \otimes e_g^{\pm} \otimes e(\chi_f) \otimes e(\chi_g)$ to a rational multiple of $(2\pi i)^{1-n} e_f^{\mp, \vee} \otimes e_g^{\mp, \vee}$. Let us define

$$\nu_f := \phi_B \circ \phi^{-1}(\omega'_f \otimes \omega(\chi_f)) = (2\pi i)^{-k-1} (\alpha_f^+ e_f^{-, \vee} + \alpha_f^- e_f^{+, \vee}) = (2\pi i)^{-k-1} (\alpha_f^+ e_f^{-, \vee} + \alpha_f^- e_f^{+, \vee})$$

and

$$\nu_g := (2\pi i)^{-\ell-1} (\alpha_g^+ e_g^{-, \vee} + \alpha_g^- e_g^{+, \vee}).$$

Also we define

$$\overline{\nu_{g^*}} = \overline{F}_\infty^*(\nu_g) = (2\pi i)^{-\ell-1} (-\alpha_g^+ e_g^{-, \vee} + \alpha_g^- e_g^{+, \vee}),$$

where \overline{F}_∞^* is the involution defined in [7, 1.4]. We define

$$\Omega := G(\overline{\chi_f}) G(\overline{\chi_g}) \nu_f \otimes \overline{\nu_{g^*}} \in H_B^{k+\ell+2}(M(f \otimes g)^\vee(1-n)) \otimes \mathbb{C} = H_B^{k+\ell+2}(M(f^* \otimes g^*)(j+1)) \otimes \mathbb{C}.$$

Since $\phi_B \circ \phi^{-1} \circ \phi_{\text{dR}}(\omega'_{f*} \otimes \overline{\omega'_g}) = \phi_B \circ \phi^{-1} \circ \phi_{\text{dR}}(\omega'_{f*} \otimes \overline{F_\infty^*}(\omega'_{g*}))$ is a $K_{f,g}^\times$ -rational multiple of $\nu_f \otimes \overline{F_\infty^*}(\nu_g) = \nu_f \otimes \overline{\nu_{g*}}$, it follows that $\phi_{\text{dR}}^{-1} \circ \phi \circ \phi_B^{-1}(\Omega)$ is a $K_{f,g}^\times$ -rational multiple of

$$G(\overline{\chi_f})G(\overline{\chi_g})\omega'_{f*} \otimes \overline{\omega'_g} = \omega_{f*} \otimes \overline{\omega_g} \in H_{\text{dR}}^{k+\ell+2}(M(f^* \otimes g^*)(j+1)) \otimes \mathbb{C}.$$

Lemma 2.1. *The map*

$$\langle \cdot, \Omega \rangle : H_B^{k+\ell+2}(M(f \otimes g)(n-1))^+ \otimes \mathbb{R} \rightarrow K_{f,g} \otimes \mathbb{C}$$

factors through $H_D^{k+\ell+3}(M(f \otimes g)(n))$.

Proof. It suffices to check that $\langle \pi(\omega'_f \otimes \omega'_g), \Omega \rangle = 0$. We have

$$\begin{aligned} \langle \pi(\omega'_f \otimes \omega'_g), \Omega \rangle &= \langle G(\chi_f)^{-1}G(\chi_g)^{-1}(\alpha_f^+ \alpha_g^{(-1)^{n+1}} e_f^+ \otimes e_g^{(-1)^{n+1}} + \alpha_f^- \alpha_g^{(-1)^n} e_f^- \otimes e_g^{(-1)^n}), \\ &\quad G(\overline{\chi_f})G(\overline{\chi_g})(\alpha_f^+ e_f^{-,\vee} + \alpha_f^- e_f^{+,\vee}) \otimes (-\alpha_g^+ e_g^{-,\vee} + \alpha_g^- e_g^{+,\vee}) \cdot (2\pi i)^{-k-\ell-2} \rangle \\ &= \left(\alpha_f^+ \alpha_g^{(-1)^{n+1}} \alpha_f^- (-1)^{n+1} \alpha_g^{(-1)^n} + \alpha_f^- \alpha_g^{(-1)^n} \alpha_f^+ (-1)^n \alpha_g^{(-1)^{n+1}} \right) \cdot \frac{G(\overline{\chi_f})G(\overline{\chi_g})}{G(\chi_f)G(\chi_g)} (2\pi i)^{-k-\ell-2} \\ &= 0. \end{aligned}$$

□

Lemma 2.2. *We have $\langle t, \Omega \rangle = (-1)^{n+1} \chi_f(-1) \chi_g(-1) N_{\chi_f} N_{\chi_g} (2\pi i)^{k+\ell-2j}$.*

Proof. We have

$$\begin{aligned} \langle t, \Omega \rangle &= \langle G(\chi_f)G(\chi_g)(2\pi i)^{2n-2}(\alpha_f^- \alpha_g^{(-1)^n})^{-1} e_f^+ \otimes e_g^{(-1)^{n+1}}, \\ &\quad G(\overline{\chi_f})G(\overline{\chi_g})(\alpha_f^+ e_f^{-,\vee} + \alpha_f^- e_f^{+,\vee}) \otimes (-\alpha_g^+ e_g^{-,\vee} + \alpha_g^- e_g^{+,\vee}) \cdot (2\pi i)^{k-\ell-2} \rangle \\ &= G(\chi_f)G(\overline{\chi_f})G(\chi_g)G(\overline{\chi_g})(2\pi i)^{k+\ell-2j}(\alpha_f^- \alpha_g^{(-1)^n})^{-1} \alpha_f^- (-1)^{n+1} \alpha_g^{(-1)^n} \\ &= (-1)^{n+1} \chi_f(-1) \chi_g(-1) N_{\chi_f} N_{\chi_g} (2\pi i)^{k+\ell-2j}, \end{aligned}$$

since for any Dirichlet character χ , we have $G(\chi)G(\overline{\chi}) = \chi(-1)N_\chi$. □

3. EISENSTEIN SYMBOLS

Here we recall Beilinson's theory of the Eisenstein symbol [3]. Let $N \geq 3$ be an integer. The complex points of E^k are given by [9, (3.4), (3.6)]

$$E^k(\mathbb{C}) \cong (\mathbb{Z}^{2k} \rtimes \text{SL}_2(\mathbb{Z})) \backslash \left(\mathcal{H} \times \mathbb{C}^k \times \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \right).$$

where the action of $\text{SL}_2(\mathbb{Z})$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\tau; z_1, \dots, z_k; h) = \left(\frac{a\tau + b}{c\tau + d}; \frac{z_1}{c\tau + d}, \dots, \frac{z_k}{c\tau + d}; \begin{pmatrix} a & b \\ c & d \end{pmatrix} h \right)$$

and the action of \mathbb{Z}^{2k} is given by

$$(u_1, v_1, \dots, u_k, v_k) \cdot (\tau; z_1, \dots, z_k; h) = (\tau; z_1 + u_1 - v_1\tau, \dots, z_k + u_k - v_k\tau; h).$$

Let ε_k be the signature character of \mathfrak{S}_{k+1} on $E^k \subset E^{k+1}$. For $i = 0, \dots, k$, let q_i denote the composition of $E^k \hookrightarrow E^{k+1} \xrightarrow{\text{pr}_i} E$. Denote $\mathcal{U}_N = E \setminus E[N]$, where $E[N]$ is the N -torsion subgroup. Write $\mathcal{U}_N^{(i)} = q_i^{-1}(\mathcal{U}_N)$ and $\mathcal{U}'_N = \bigcap_{i=0}^k \mathcal{U}_N^{(i)} \subset E^k$.

Choose $g_0, \dots, g_k \in \mathcal{O}(\mathcal{U}_N)^\times$. Denote $g = q_0^*(g_0) \cup \dots \cup q_k^*(g_k) \in H_{\mathcal{M}}^{k+1}(\mathcal{U}'_N, \mathbb{Q}(k+1))$. Write $\tilde{G} = (\mathbb{Z}/N\mathbb{Z})^{2k} \rtimes \mathfrak{S}_{k+1}$. Here $(\mathbb{Z}/N\mathbb{Z})^{2k} \simeq E[N]^k$ acts on $E[N]$ by the natural translation. Let $\varepsilon_k : \tilde{G} \rightarrow \{\pm 1\}$ be the signature character defined by $\varepsilon(g) = \varepsilon(\sigma) = \text{sign}(\sigma)$ for $g = (t, \sigma) \in \tilde{G} = (\mathbb{Z}/N\mathbb{Z})^{2k} \rtimes \mathfrak{S}_{k+1}$. Then \tilde{G} acts on E^k and \mathcal{U}'_N . This induces the action of \tilde{G} on the motivic cohomology $H_{\mathcal{M}}^{k+1}(\mathcal{U}'_N, \mathbb{Q}(k+1))$. Denote the idempotent corresponding to ε_k by \tilde{e}_k . Hence we have the \tilde{e}_k -eigenspace $H_{\mathcal{M}}^{k+1}(\mathcal{U}'_N, \mathbb{Q}(k+1))^{\tilde{e}_k}$ and the projection

$$\text{pr}_{\tilde{e}_k} : H_{\mathcal{M}}^{k+1}(\mathcal{U}'_N, \mathbb{Q}(k+1)) \rightarrow H_{\mathcal{M}}^{k+1}(\mathcal{U}'_N, \mathbb{Q}(k+1))^{\tilde{e}_k}$$

defined by $x \mapsto |\tilde{G}|^{-1} \sum_{g \in \tilde{G}} \varepsilon_k(g) g \cdot x$.

Let M be a positive auxiliary integer. Let $j : \mathcal{U}'_{MN} \hookrightarrow \mathcal{U}'_N$ be the canonical inclusion and $[\times M] : \mathcal{U}'_{MN} \rightarrow \mathcal{U}'_N$ the multiplication by M . Then j and $[\times M]$ induce

$$j^* : H_{\mathcal{M}}^{k+1}(\mathcal{U}'_N, \mathbb{Q}(k+1)) \rightarrow H_{\mathcal{M}}^{k+1}(\mathcal{U}'_{MN}, \mathbb{Q}(k+1))^{(\mathbb{Z}/M\mathbb{Z})^{2k}}$$

and

$$[\times M]^* : H_{\mathcal{M}}^{k+1}(\mathcal{U}'_N, \mathbb{Q}(k+1)) \xrightarrow{\sim} H_{\mathcal{M}}^{k+1}(\mathcal{U}'_{MN}, \mathbb{Q}(k+1))^{(\mathbb{Z}/M\mathbb{Z})^{2k}}.$$

Write $[\times M^{-1}] = ([\times M]^*)^{-1} \circ j^*$. Denote by $H_{\mathcal{M}}^{k+1}(\mathcal{U}'_N, \mathbb{Q}(k+1))_{\tilde{e}_k}^{\tilde{e}_k}$ the maximal quotient of $H_{\mathcal{M}}^{k+1}(\mathcal{U}'_N, \mathbb{Q}(k+1))_{\tilde{e}_k}^{\tilde{e}_k}$ such that $[\times M^{-1}] = M^{-k}$ for any $M \geq 1$. Then we have a canonical projection

$$\overline{\text{pr}}_{\tilde{e}_k} : H_{\mathcal{M}}^{k+1}(\mathcal{U}'_N, \mathbb{Q}(k+1)) \rightarrow H_{\mathcal{M}}^{k+1}(\mathcal{U}'_N, \mathbb{Q}(k+1))_{\tilde{e}_k}^{\tilde{e}_k}.$$

Theorem 3.1 ([8, (8.16) Theorem]). *The canonical map*

$$\alpha^* : H_{\mathcal{M}}^{k+1}(E^k, \mathbb{Q}(k+1))^{e_k} = H_{\mathcal{M}}^{k+1}(E^k, \mathbb{Q}(k+1))_{\tilde{e}_k}^{\tilde{e}_k} \rightarrow H_{\mathcal{M}}^{k+1}(\mathcal{U}'_N, \mathbb{Q}(k+1))_{\tilde{e}_k}^{\tilde{e}_k}$$

induced by $\alpha : \mathcal{U}'_N \hookrightarrow E^k$ is bijective.

We write $\widetilde{\text{Eis}}^k(g_0, \dots, g_k) = (\alpha^*)^{-1}(\overline{\text{pr}}_{\tilde{e}_k}(g)) \in H_{\mathcal{M}}^{k+1}(E^k, \mathbb{Q}(k+1))$ for $g_0, \dots, g_k \in \mathcal{O}(\mathcal{U}_N)^\times$. In fact, $\widetilde{\text{Eis}}^k$ factors through the divisors $\mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^2]^0$. Therefore we have a commutative diagram:

$$\begin{array}{ccc} \bigotimes_{i=0}^k \mathcal{O}(\mathcal{U}_N)^\times & \xrightarrow{\widetilde{\text{Eis}}^k} & H_{\mathcal{M}}^{k+1}(E^k, \mathbb{Q}(k+1)) \\ \text{Div} \downarrow & & \uparrow \text{Eis}^k \\ \bigotimes_{i=0}^k \mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^2]_{\varepsilon_k}^0 & \xleftarrow[\simeq]{\theta} & \mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^2]^0 \end{array}$$

where θ is defined by $\beta \mapsto [\beta \otimes \alpha \otimes \dots \otimes \alpha]$ with $\alpha = N^2[0] - \sum_{x \in (\mathbb{Z}/N\mathbb{Z})^2} [x]$. The map

$$\text{Eis}^k : \mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^2]^0 \rightarrow H_{\mathcal{M}}^{k+1}(E^k, \mathbb{Q}(k+1))^{e_k}$$

is called the Eisenstein symbol. For a smooth projective variety X over \mathbb{R} , let $H_{\mathcal{D}}^i(X, \mathbb{R}(j))$ denote its Deligne cohomology.

We now recall an explicit formula for the realization of the Eisenstein symbol. Fix an integer $k \geq 0$. Let

$$r_{\mathcal{D}} : H_{\mathcal{M}}^{k+1}(E^k, \mathbb{Q}(k+1))^{e_k} \rightarrow H_{\mathcal{D}}^{k+1}(E_{\mathbb{R}}^k, \mathbb{R}(k+1))^{e_k}$$

be the regulator map.

By [19, (7.3)], the Deligne cohomology group is given by:

$$H_{\mathcal{D}}^{k+1}(E_{\mathbb{R}}^k, \mathbb{R}(k+1)) \simeq \frac{\{\varphi \in H^0(E_{\mathbb{R}, \text{an}}^k, \mathcal{A}^k \otimes \mathbb{R}(k)) \mid d\varphi = \frac{1}{2}(\omega + (-1)^k \bar{\omega}), \omega \in \Omega^{k+1}(\overline{E}^k) \langle D \rangle\}}{dH^0(E_{\mathbb{R}, \text{an}}^k, \mathcal{A}^{k-1} \otimes \mathbb{R}(k))},$$

where \mathcal{A} is the de Rham complex of real valued C^∞ -forms, \overline{E}^k is a smooth compactification of $E^k(\mathbb{C})$ and $D = \overline{E}^k \setminus E^k(\mathbb{C})$.

Recall that

$$E^k(\mathbb{C}) \cong (\mathbb{Z}^{2k} \rtimes \text{SL}_2(\mathbb{Z})) \backslash (\mathcal{H} \times \mathbb{C}^k \times \text{GL}_2(\mathbb{Z}/N\mathbb{Z})).$$

Write τ (resp. z_1, \dots, z_k) for the coordinate on \mathcal{H} (resp. \mathbb{C}^k). Write h for an element of $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$. For any integer $0 \leq j \leq k$, define

$$\psi_{k,j} = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \varepsilon(\sigma) d\bar{z}_{\sigma(1)} \wedge \dots \wedge d\bar{z}_{\sigma(j)} \wedge dz_{\sigma(j+1)} \wedge \dots \wedge dz_{\sigma(k)}.$$

Let $\beta \in \mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^2]^0$. Then by [9, (3.12), (3.28)] and [12, Remark after Lemma 7.1], $r_{\mathcal{D}}(\text{Eis}^k(\beta))$ is represented by

$$\Phi^k(\beta) := -\frac{k!(k+2)}{N(2\pi i)} \cdot \frac{\tau - \bar{\tau}}{2} \sum_{a=0}^k \psi_{k,a} \cdot \left(\sum'_{(c,d) \in \mathbb{Z}^2} \sum_{v \in (\mathbb{Z}/N\mathbb{Z})^2} \frac{\beta(h^{-1}v) \cdot e^{\frac{2\pi i(c v_1 + d v_2)}{N}}}{(c\tau + d)^{k+1-a} (c\bar{\tau} + d)^{a+1}} \right) \pmod{d\tau, d\bar{\tau}}$$

where \sum' denotes that we omit the term $(c, d) = (0, 0)$. For brevity, for any $a, b \geq 1$ we put

$$\mathcal{E}_\beta^{a,b}(\tau, h) := \sum'_{(c,d) \in \mathbb{Z}^2} \sum_{v \in (\mathbb{Z}/N\mathbb{Z})^2} \frac{\beta(h^{-1}v) \cdot e^{\frac{2\pi i(cu_1 + dv_2)}{N}}}{(c\tau + d)^a (c\bar{\tau} + d)^b}.$$

4. CONSTRUCTION OF ELEMENTS IN THE MOTIVIC COHOMOLOGY

Let k, ℓ be non-negative integers with $k \leq \ell$ and choose an integer j such that $0 \leq j \leq k$. Write $k' = k - j \geq 0$ and $\ell' = \ell - j \geq 0$. Consider the following three morphisms:

(1) $p : E^{k'+j+\ell'} \rightarrow E^{k'+\ell'}$ given by

$$(\tau; u_1, \dots, u_{k'}, t_1, \dots, t_j, v_1, \dots, v_{\ell'}; h) \mapsto (\tau; u_1, \dots, u_{k'}, v_1, \dots, v_{\ell'}; h).$$

(2) $\Delta : E^{k'+j+\ell'} \rightarrow E^{k'+2j+\ell'} = E^{k+\ell}$ given by

$$(\tau; u_1, \dots, u_{k'}, t_1, \dots, t_j, v_1, \dots, v_{\ell'}; h) \mapsto (\tau; u_1, \dots, u_{k'}, t_1, \dots, t_j, t_1, \dots, t_j, v_1, \dots, v_{\ell'}; h).$$

(3) $i : E^{k'+2j+\ell'} = E^{k+\ell} \rightarrow E^k \times E^\ell$ given by

$$(\tau; u_1, \dots, u_{k'}, t_1, \dots, t_j, t'_1, \dots, t'_j, v_1, \dots, v_{\ell'}; h) \mapsto ((\tau; u_1, \dots, u_{k'}, t_1, \dots, t_j; h), (\tau; t'_1, \dots, t'_j, v_1, \dots, v_{\ell'}; h)).$$

Note that $(i \circ \Delta)(\tau; u, t, v; h) = ((\tau; u, t; h), (\tau; t, v; h))$.

Definition 4.1. For $\beta \in \mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^2]^0$, denote by $\Xi^{k,\ell,j}(\beta)$ the image of $\text{Eis}^{k'+\ell'}(\beta)$ under the composite of morphisms:

$$\begin{aligned} H_{\mathcal{M}}^{k'+\ell'+1}(E^{k'+\ell'}, \mathbb{Q}(k' + \ell' + 1)) &\xrightarrow{p^*} H_{\mathcal{M}}^{k'+\ell'+1}(E^{k'+j+\ell'}, \mathbb{Q}(k' + \ell' + 1)) \\ &\xrightarrow{\Delta_*} H_{\mathcal{M}}^{k+\ell+1}(E^{k+\ell}, \mathbb{Q}(k + \ell - j + 1)) \\ &\xrightarrow{i_*} H_{\mathcal{M}}^{k+\ell+3}(E^k \times E^\ell, \mathbb{Q}(k + \ell - j + 2)). \end{aligned}$$

5. THE RANKIN-SELBERG METHOD

Let $L(f \otimes g, s)$ denote the L -function associated to the 4-dimensional Galois representation $V_f \otimes V_g$. We have

$$L(f \otimes g, s) = \prod_{p \text{ prime}} P_p(f \otimes g, s)^{-1},$$

where $P_p(f \otimes g, s) = \det(1 - \text{Frob}_p \cdot p^{-s} | (V_f \otimes V_g)^{I_p})$ is a polynomial in p^{-s} . Then the polynomial $P_p(f \otimes g, s)$ coincides up to the shift $s \mapsto s - \frac{k+\ell+2}{2}$ with the automorphic L -factor defined by Jacquet in [13], and $L(f \otimes g, s)$ converges for $\text{Re}(s) > \frac{k+\ell}{2} + 2$.

Let N be an integer divisible by N_f and N_g . Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ be the Dirichlet character induced by $\chi_f \chi_g$. Put $D(f, g, s) := \sum_{n=1}^\infty a_n(f) a_n(g) n^{-s}$. By [23, Lemma 1], we have

$$L(\chi, 2s - k - \ell - 2) D(f, g, s) = R_{f,g,N}(s) L(f \otimes g, s),$$

where

$$R_{f,g,N}(s) := \left(\prod_{p|N} P_p(f \otimes g, s) \right) \sum_{n \in S(N)} \frac{a_n(f) a_n(g)}{n^s}$$

is a polynomial in the variables p^{-s} for $p|N$ by [13, Theorem 15.1]. Here $S(N)$ denotes the set of integers all of whose prime factors divide N .

For any Dirichlet character $\omega : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$, define the Eisenstein series

$$E_{\ell-k,N}(\tau, s, \omega) = \sum'_{m,n \in \mathbb{Z}} \frac{\omega(n)}{(Nm\tau + n)^{\ell-k} |Nm\tau + n|^{2s}}.$$

Theorem 5.1 (Shimura [23, (2.4)]). *We have*

$$\int_{\Gamma_0(N) \backslash \mathcal{H}} f(\tau) g(-\bar{\tau}) E_{\ell-k,N}(\tau, s-1-\ell, \chi) y^{s-1} dx dy = 2(4\pi)^{-s} \Gamma(s) L(\chi, 2s - k - \ell - 2) D(f, g, s).$$

Remark 5.2. Let us assume $k = \ell$. Then by [23, (2.5)] and [24, page 220, Correction], $D(f, g, s)$ has a pole at $s = k + 2$ if and only if $\langle f^*, g \rangle \neq 0$. This is equivalent to $g = f^*$. In this case, we have $\chi_g = \chi_f^{-1}$, hence χ is trivial. Therefore, our assumption $\chi \neq 1$ excludes the case where $L(f \otimes g, s)$ has a pole.

6. COMPUTATION OF THE REGULATOR INTEGRAL

Let j be an integer satisfying $0 \leq j \leq k \leq \ell$. Recall that we have a differential form $\Omega_{f,g} := \omega_{f^*} \otimes \overline{\omega_g} \in H_{\text{dR}}^{k+\ell+2}(M(f^* \otimes g^*)(j+1)) \otimes \mathbb{C}$. Since $M(f^* \otimes g^*)$ is a direct factor of $h^{k+\ell+2}(E^k \times E^\ell) \otimes K_{f,g}$, we may consider $\Omega_{f,g}$ as an element of $H_{\text{dR}}^{k+\ell+2}(E^k \times E^\ell) \otimes K_{f,g} \otimes \mathbb{C}$. By the same argument as in Lemma 2.1, and since $\Omega_{f,g}$ has rapid decay at infinity, pairing with $\Omega_{f,g}$ yields a linear map

$$\langle \cdot, \Omega_{f,g} \rangle : H_{\mathcal{D}}^{k+\ell+3}(E_{\mathbb{R}}^k \times E_{\mathbb{R}}^\ell, \mathbb{R}(n)) \rightarrow K_{f,g} \otimes \mathbb{C}.$$

Let $\beta \in \mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^2]^0$. In this section, we compute $\langle r_{\mathcal{D}}(\Xi^{k,\ell,j}(\beta)), \Omega_{f,g} \rangle$ in terms of the Rankin-Selberg L -function of f and g . At the beginning β is arbitrary, but from Definition 6.2 on, we will use a particular choice of β .

Lemma 6.1. *We have*

$$\langle r_{\mathcal{D}}(\Xi^{k,\ell,j}(\beta)), \Omega_{f,g} \rangle = \frac{1}{(2\pi i)^{k'+j+\ell'+1}} \int_{E^{k'+j+\ell'}(\mathbb{C})} p^* \Phi^{k'+\ell'}(\beta) \wedge \Delta^* i^* \Omega_{f,g}.$$

Proof. We have

$$\begin{aligned} \langle r_{\mathcal{D}}(\Xi^{k,\ell,j}(\beta)), \Omega_{f,g} \rangle &= \langle r_{\mathcal{D}}(i_* \Delta_* p^* \text{Eis}^{k'+\ell'}(\beta)), \Omega_{f,g} \rangle \\ &= \langle r_{\mathcal{D}}(p^* \text{Eis}^{k'+\ell'}(\beta)), \Delta^* i^* \Omega_{f,g} \rangle \\ &= \frac{1}{(2\pi i)^{\dim E^{k'+j+\ell'}}} \int_{E^{k'+j+\ell'}(\mathbb{C})} p^* r_{\mathcal{D}}(\text{Eis}^{k'+\ell'}(\beta)) \wedge \Delta^* i^* \Omega_{f,g} \\ &= \frac{1}{(2\pi i)^{k'+j+\ell'+1}} \int_{E^{k'+j+\ell'}(\mathbb{C})} p^* \Phi^{k'+\ell'}(\beta) \wedge \Delta^* i^* \Omega_{f,g}. \end{aligned}$$

□

Let $\tau, z_1, \dots, z_{k+\ell-j}$ denote the coordinates on $E^{k+\ell-j}(\mathbb{C})$. Note that the differential form

$$\Delta^* i^* \Omega_{f,g} = (-1)^{k+\ell+1} (2\pi i)^{k+\ell+2} f^*(\tau) \overline{g(\tau)} d\tau \wedge d\overline{\tau} \wedge dz_1 \wedge \dots \wedge dz_k \wedge d\overline{z}_{k-j+1} \wedge \dots \wedge d\overline{z}_{k+\ell-j}$$

already contains $d\tau \wedge d\overline{\tau}$. Therefore, we may neglect the terms of $\Phi^{k'+\ell'}(\beta)$ involving $d\tau, d\overline{\tau}$. Moreover, we have

$$p^* \psi_{k'+\ell',a} \wedge \Delta^* i^* \Omega_{f,g} = \begin{cases} C_1 f^*(\tau) \overline{g(\tau)} d\tau \wedge d\overline{\tau} \wedge \bigwedge_{i=1}^{k+\ell-j} dz_i \wedge d\overline{z}_i & \text{if } a = k', \\ 0 & \text{if } a \neq k', \end{cases}$$

with

$$C_1 = (-1)^{k+\ell+1+k'^2+j(k'+\ell')+(k'+j+\ell')(k'+j+\ell'-1)/2} \frac{k'! \cdot \ell'!}{(k'+\ell')!} (2\pi i)^{k+\ell+2}.$$

It follows that

$$\begin{aligned} & p^* \Phi^{k'+\ell'}(\beta) \wedge \Delta^* i^* \Omega_{f,g} \\ &= - \frac{(k'+\ell')! \cdot (k'+\ell'+2)}{N(2\pi i)} \cdot \frac{\tau - \overline{\tau}}{2} \cdot \mathcal{E}_{\beta}^{\ell'+1,k'+1}(\tau, h) \cdot p^* \psi_{k'+\ell',k'} \wedge \Delta^* i^* \Omega_{f,g} \\ &= - \frac{C_1 \cdot (k'+\ell')! \cdot (k'+\ell'+2)}{N(2\pi i)} \cdot \frac{\tau - \overline{\tau}}{2} \cdot \mathcal{E}_{\beta}^{\ell'+1,k'+1}(\tau, h) \cdot f^*(\tau) \overline{g(\tau)} d\tau \wedge d\overline{\tau} \wedge \bigwedge_{i=1}^{k+\ell-j} dz_i \wedge d\overline{z}_i. \end{aligned}$$

Recall [9, (3.4)] that the complex points of $Y(N)$ are given by

$$Y(N)(\mathbb{C}) = \text{SL}_2(\mathbb{Z}) \backslash (\mathcal{H} \times \text{GL}_2(\mathbb{Z}/N\mathbb{Z})).$$

Note that $\int_{\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})} dz \wedge d\bar{z} = -2i\text{Im}(\tau)$. Using Lemma 6.1 and integrating over the fibers of $E^{k'+j+\ell'}$ over $Y(N)$, we get

$$\langle r_{\mathcal{D}}(\Xi^{k,\ell,j}(\beta)), \Omega_{f,g} \rangle = -\frac{(-2i)^{k+\ell-j} \cdot i \cdot C_1 \cdot (k' + \ell' + 2)!}{(2\pi i)^{k+\ell-j+2} \cdot N \cdot (k' + \ell' + 1)} \int_{Y(N)(\mathbb{C})} f^*(\tau) \overline{g(\tau)} \mathcal{E}_{\beta}^{\ell'+1, k'+1}(\tau, h) \text{Im}(\tau)^{k+\ell-j+1} d\tau \wedge d\bar{\tau}.$$

We have an isomorphism of analytic spaces

$$\begin{aligned} \nu : (\mathbb{Z}/N\mathbb{Z})^{\times} \times \Gamma(N) \backslash \mathcal{H} &\xrightarrow{\cong} Y(N)(\mathbb{C}) \\ (a, [\tau]) &\mapsto \left[\left(\tau, \begin{pmatrix} 0 & -1 \\ a & 0 \end{pmatrix} \right) \right]. \end{aligned}$$

Note that $\nu(a, \tau)$ corresponds to the elliptic curve $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ with basis of N -torsion $(a\tau/N, 1/N)$ in the moduli space.

Let $\chi : (\mathbb{Z}/N\mathbb{Z})^{\times} \rightarrow \mathbb{C}^{\times}$ be the Dirichlet character induced by $\chi_f \chi_g$. Assume $\chi \neq 1$.

Definition 6.2. Let $\beta_{\chi} \in \mathbb{Q}(\chi)[(\mathbb{Z}/N\mathbb{Z})^2]^0 \subset K_{f,g}[(\mathbb{Z}/N\mathbb{Z})^2]^0$ be the divisor defined by

$$\beta_{\chi}(v_1, v_2) = \begin{cases} \overline{\chi}(-v_2) & \text{if } v_1 = 0, \\ 0 & \text{if } v_1 \neq 0. \end{cases}$$

For an integer $w \geq 0$, $\alpha \in \mathbb{Q}/\mathbb{Z}$, $\tau \in \mathcal{H}$ and $s \in \mathbb{C}$, define the following standard real-analytic Eisenstein series as in [18, Definition 4.2.1]:

$$E_{\alpha}^{(w)}(\tau, s) = (-2\pi i)^{-w} \pi^{-s} \Gamma(s+w) \sum'_{m,n \in \mathbb{Z}} \frac{\text{Im}(\tau)^s}{(m\tau + n + \alpha)^w |m\tau + n + \alpha|^{2s}},$$

where \sum' denotes that the term $(m, n) = (0, 0)$ is omitted if $\alpha = 0$, and

$$F_{\alpha}^{(w)}(\tau, s) = (-2\pi i)^{-w} \pi^{-s} \Gamma(s+w) \sum'_{m,n \in \mathbb{Z}} \frac{e^{2\pi i \alpha m} \text{Im}(\tau)^s}{(m\tau + n)^w |m\tau + n|^{2s}},$$

where \sum' denotes that the term $(m, n) = (0, 0)$ is omitted. For fixed w, α, τ , these functions have meromorphic continuations to the whole s -plane, and are holomorphic everywhere if $w \neq 0$. Note that

$$\sum_{\alpha \in (\mathbb{Z}/N\mathbb{Z})^{\times}} \omega(\alpha) E_{\alpha/N}^{(\ell-k)}(\tau, s) = (-2\pi i)^{-\ell+k} \pi^{-s} \Gamma(s+\ell-k) \text{Im}(\tau)^s N^{\ell-k+2s} E_{\ell-k,N}(\tau, s, \omega).$$

Lemma 6.3. For any $a \in (\mathbb{Z}/N\mathbb{Z})^{\times}$, we have

$$(6.2) \quad \mathcal{E}_{\beta_{\chi}}^{\ell'+1, k'+1} \left(\tau, \begin{pmatrix} 0 & -1 \\ a & 0 \end{pmatrix} \right) = \frac{\pi^{k'+\ell'+1}}{\ell'! N^{k'+\ell'} \cdot \text{Im}(\tau)^{k'+\ell'+1}} \lim_{s \rightarrow -\ell'} \Gamma(s+\ell-k) E_{\ell-k,N}(\tau, s, \overline{\chi}).$$

Proof. We have

$$\begin{aligned} \mathcal{E}_{\beta_{\chi}}^{\ell'+1, k'+1} \left(\tau, \begin{pmatrix} 0 & -1 \\ a & 0 \end{pmatrix} \right) &= \sum'_{(c,d) \in \mathbb{Z}^2} \sum_{v_1, v_2 \in \mathbb{Z}/N\mathbb{Z}} \frac{\beta_{\chi}(a^{-1}v_2, -v_1) \cdot e^{\frac{2\pi i(c v_1 + d v_2)}{N}}}{(c\tau + d)^{\ell'+1} (c\bar{\tau} + d)^{k'+1}} \\ &= \sum'_{(c,d) \in \mathbb{Z}^2} \sum_{v_1 \in (\mathbb{Z}/N\mathbb{Z})^{\times}} \frac{\overline{\chi}(v_1) \cdot e^{\frac{2\pi i c v_1}{N}}}{(c\tau + d)^{\ell'+1} (c\bar{\tau} + d)^{k'+1}} \\ &= \frac{(-2\pi i)^{\ell-k} \pi^{k'+1}}{\ell'! \cdot \text{Im}(\tau)^{k'+1}} \sum_{v_1 \in (\mathbb{Z}/N\mathbb{Z})^{\times}} \overline{\chi}(v_1) F_{v_1/N}^{(\ell-k)}(\tau, k'+1) \\ &= \frac{(-2\pi i)^{\ell-k} \pi^{k'+1}}{\ell'! \cdot \text{Im}(\tau)^{k'+1}} \sum_{v_1 \in (\mathbb{Z}/N\mathbb{Z})^{\times}} \overline{\chi}(v_1) E_{v_1/N}^{(\ell-k)}(\tau, -\ell') \quad \text{by [18, 4.2.2(iv)]} \\ &= \frac{\pi^{k'+\ell'+1}}{\ell'! N^{k'+\ell'} \cdot \text{Im}(\tau)^{k'+\ell'+1}} \lim_{s \rightarrow -\ell'} \Gamma(s+\ell-k) E_{\ell-k,N}(\tau, s, \overline{\chi}). \end{aligned}$$

□

Note that the right hand side of (6.2) is independent of $a \in (\mathbb{Z}/N\mathbb{Z})^\times$. Therefore, the contributions of the regulator integral over each connected component of $Y(N)(\mathbb{C})$ are equal, and we have

$$\langle r_{\mathcal{D}}(\Xi^{k,\ell,j}(\beta_\chi)), \Omega_{f,g} \rangle = \frac{C_2 \cdot \phi(N)}{(2\pi i)^{k+\ell-j+1}} \int_{\Gamma(N) \setminus \mathcal{H}} f^*(\tau) g^*(-\bar{\tau}) \text{Im}(\tau)^j \lim_{s \rightarrow -\ell'} \Gamma(s + \ell - k) E_{\ell-k,N}(\tau, s, \bar{\chi}) dx dy$$

with

$$C_2 = \frac{(-2i)^{k+\ell-j} \cdot i \cdot \pi^{k'+\ell'} \cdot (k' + \ell' + 2)!}{N^{k'+\ell'+1} \cdot \ell'! \cdot (k' + \ell' + 1)} \cdot C_1.$$

Since the integrand is invariant under the group $\Gamma_0(N)$, this can be rewritten as

$$\langle r_{\mathcal{D}}(\Xi^{k,\ell,j}(\beta_\chi)), \Omega_{f,g} \rangle = \frac{C_2 \cdot N \cdot \phi(N)^2}{2(2\pi i)^{k+\ell-j+1}} \lim_{s \rightarrow -\ell'} \Gamma(s + \ell - k) \int_{\Gamma_0(N) \setminus \mathcal{H}} f^*(\tau) g^*(-\bar{\tau}) E_{\ell-k,N}(\tau, s, \bar{\chi}) y^{s+\ell} dx dy.$$

Using Theorem 5.1 with f^* and g^* , we get

$$\begin{aligned} \langle r_{\mathcal{D}}(\Xi^{k,\ell,j}(\beta_\chi)), \Omega_{f,g} \rangle &= \frac{C_2 \cdot N \cdot \phi(N)^2}{2(2\pi i)^{k+\ell-j+1}} \lim_{s \rightarrow -\ell'} \Gamma(s + \ell - k) \cdot 2 \cdot (4\pi)^{-s-1-\ell} \Gamma(s + 1 + \ell) \\ &\quad \cdot R_{f^*,g^*,N}(s + 1 + \ell) L(f^* \otimes g^*, s + 1 + \ell) \\ &= \frac{C_2 \cdot N \cdot \phi(N)^2}{(2\pi i)^{k+\ell-j+1}} (4\pi)^{-j-1} \cdot j! \cdot R_{f^*,g^*,N}(j + 1) \lim_{s \rightarrow -\ell'} \Gamma(s + \ell - k) L(f^* \otimes g^*, s + 1 + \ell) \\ &= \frac{C_2 \cdot N \cdot \phi(N)^2}{(2\pi i)^{k+\ell-j+1}} (4\pi)^{-j-1} \cdot j! \cdot R_{f^*,g^*,N}(j + 1) \frac{(-1)^{k-j}}{(k-j)!} L'(f^* \otimes g^*, j + 1). \end{aligned}$$

Putting everything together, we have the following theorem.

Theorem 6.4. *Let $\Omega_{f,g} = \omega_{f^*} \otimes \overline{\omega_g}$. Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ be the Dirichlet character induced by $\chi_f \chi_g$. Assume $\chi \neq 1$. Then we have the following identity in $K_{f,g} \otimes \mathbb{C}$*

$$\langle r_{\mathcal{D}}(\Xi^{k,\ell,j}(\beta_\chi)), \Omega_{f,g} \rangle = \pm (2\pi i)^{k+\ell-2j} \cdot \frac{(k + \ell - 2j + 2) \cdot j! \cdot \phi(N)^2}{2 \cdot N^{k+\ell-2j}} \cdot R_{f^*,g^*,N}(j + 1) \cdot L'(f^* \otimes g^*, j + 1).$$

Note that $R_{f^*,g^*,N}(j + 1)$ is an element of $K_{f,g}$.

7. COMPUTATION OF RESIDUES

In this section, we extend the motivic element $\Xi^{k,\ell,j}(\beta)$ to the Néron model by computing the residue.

7.1. Voevodsky's category of motives and motivic cohomology. For a field k , let $DM_{gm}^{\text{eff}}(k)$ be the category of effective geometrical motives over k . For a scheme X over k , we have the motive $M_{gm}(X)$ and the motive with compact support $M_{gm}^c(X)$. We consider the \mathbb{Q} -linear analogue of $DM_{gm}^{\text{eff}}(k)$ denoted by $DM_{gm}^{\text{eff}}(k)_{\mathbb{Q}}$. For any object M of $DM_{gm}^{\text{eff}}(k)_{\mathbb{Q}}$, we define the motivic cohomology by

$$H_{\mathcal{M}}^i(M, \mathbb{Q}(j)) = \text{Hom}_{DM_{gm}^{\text{eff}}(k)_{\mathbb{Q}}}(M, \mathbb{Q}(j)[i]).$$

Then it is known that

$$H_{\mathcal{M}}^i(M_{gm}(X), \mathbb{Q}(j)) \simeq H_{\mathcal{M}}^i(X, \mathbb{Q}(j)) \simeq CH^j(X, 2j - i)$$

for a smooth separated scheme X over k , where $CH^n(X, m)$ is Bloch's higher Chow group.

7.2. Motives for Kuga-Sato varieties. Let $Y = Y(N)$ and $X = X(N)$. Denote $X^\infty = X \setminus Y$. The symmetric group \mathfrak{S}_k acts on \overline{E}^k by permutation, $(\mathbb{Z}/N\mathbb{Z})^{2k}$ by translations, and μ_2^k by inversion in the fiber. Therefore we have the action of $G = ((\mathbb{Z}/N\mathbb{Z})^2 \rtimes \mu_2)^k \rtimes \mathfrak{S}_k$. This action can be extended to $\overline{\overline{E}}^k$. Let $\varepsilon_k : G \rightarrow \{\pm 1\}$ be the character which is trivial on $(\mathbb{Z}/N\mathbb{Z})^{2k}$, is the product on μ_2^k , and is the sign character on \mathfrak{S}_k . Then define the idempotent

$$e_k := \frac{1}{(2N^2)^k \cdot k!} \sum_{g \in G} \varepsilon_k(g)^{-1} \cdot g \in \mathbb{Z}[\frac{1}{2N \cdot k!}][G].$$

Let $M_{gm}(\overline{\overline{E}}^k)^{e_k} \in DM_{gm}^{\text{eff}}(\mathbb{Q})_{\mathbb{Q}}$ be the image of the idempotent e_k on $M_{gm}(\overline{\overline{E}}^k)$. Also denote by $M_{gm}(E^k)^{e_k}$ and $M_{gm}^c(E^k)^{e_k}$ the images of e_k on $M_{gm}(E^k)$ and $M_{gm}^c(E^k)$ respectively. Write the complement of E^k in the smooth proper scheme $\overline{\overline{E}}^k$ by $\overline{\overline{E}}^{k,\infty}$.

Now we recall a result of Schappacher-Scholl [20]. Fix an integer $N \geq 3$ and an integer $k \geq 0$. Recall $X = X(N)$ is the compactified modular curve of level N and $\overline{E} \rightarrow X$ the universal generalized elliptic curve over X . Consider the k -fold fiber product $\overline{E}^k = \overline{E} \times_X \cdots \times_X \overline{E}$ of \overline{E} over X . Denote $X^\infty = X \setminus Y$, where $Y = Y(N)$ is the modular curve of level N . Let \hat{E}^k be the Néron model of E^k over X and $\overline{\overline{E}}^k \rightarrow \overline{E}^k$ Deligne's desingularization. Then $\overline{\overline{E}}^k$ is a smooth projective variety over \mathbb{Q} .

By the generalized Gysin distinguished triangle

$$M_{gm}(E^k)^{e_k} \rightarrow M_{gm}(\overline{\overline{E}}^k)^{e_k} \rightarrow M_{gm}^c(\overline{\overline{E}}^{k,\infty})^{e_k}(1)[2] \rightarrow M_{gm}(E^k)^{e_k}[1],$$

we get the localization sequence for the pair $(\overline{\overline{E}}^k, E^k)$:

$$0 \rightarrow H_{\mathcal{M}}^{k+1}(\overline{\overline{E}}^k, \mathbb{Q}(k+1))^{e_k} \rightarrow H_{\mathcal{M}}^{k+1}(E^k, \mathbb{Q}(k+1))^{e_k} \xrightarrow{\text{Res}^k} \mathcal{F}_N^k \rightarrow 0,$$

where $\mathcal{F}_N^k \simeq H_{\mathcal{M}}^k(\overline{\overline{E}}^{k,\infty}, \mathbb{Q}(k))^{e_k} \simeq H_{\mathcal{M}}^0(X^\infty, \mathbb{Q}(0)) \simeq \mathbb{Q}[X^\infty]$ is defined by

$$\mathcal{F}_N^k = \left\{ f : \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \rightarrow \mathbb{Q} \mid f\left(g \cdot \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right) = (-1)^k f(-g) \text{ for all } a \in (\mathbb{Z}/N\mathbb{Z})^\times \text{ and } b \in \mathbb{Z}/N\mathbb{Z} \right\}.$$

Then Res^k is $\text{GL}_2(\mathbb{Z}/N\mathbb{Z})$ -equivariant. Define the horospherical map $\omega_N^k : \mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^2]^0 \rightarrow \mathcal{F}_N^k$ by

$$\omega_N^k(\beta)(g) = \sum_{x=(x_1, x_2) \in (\mathbb{Z}/N\mathbb{Z})^2} \beta(g \cdot {}^t x) B_{k+2} \left(\left\langle \frac{x_2}{N} \right\rangle \right)$$

for $\beta \in \mathbb{Q}[(\mathbb{Z}/N\mathbb{Z})^2]^0$ and $g \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z})$, where B_k is Bernoulli polynomial.

Theorem 7.1 (Schappacher-Scholl [20, 7.2]). *$\text{Res}^k \circ \text{Eis}^k$ is a nonzero multiple of ω_N^k .*

We denote $Z^k = \hat{E}^{k,*} \setminus E^k = \hat{E}^{k,*} \times_X X^\infty \simeq \mathbb{G}_m^k \times_{\mathbb{Q}} X^\infty$ (non-canonically), $E^{k,\ell} = E^k \times E^\ell$, $\hat{E}^{k,\ell,*} = \hat{E}^{k,*} \times \hat{E}^{\ell,*}$, $Z^{k,\ell} = Z^k \times Z^\ell$ and $U^{k,\ell} = \hat{E}^{k,\ell,*} \setminus Z^{k,\ell}$. Let $i' : E^{k+\ell} \rightarrow U^{k,\ell}$ be the canonical closed immersion. Then i' induces the morphism

$$i'_* : H_{\mathcal{M}}^{k+\ell+1}(E^{k+\ell}, \mathbb{Q}(k+\ell-j+1)) \rightarrow H_{\mathcal{M}}^{k+\ell+3}(U^{k,\ell}, \mathbb{Q}(k+\ell-j+2)).$$

Recall that we defined the morphisms:

$$\begin{array}{ccccc} E^{k+\ell-j} & \xrightarrow{\Delta} & E^{k+\ell} & \xrightarrow{i} & E^k \times E^\ell \\ \downarrow p & & & & \\ E^{k+\ell-2j} & & & & \end{array}$$

Similarly we define the morphisms

$$\begin{array}{ccccc} \hat{E}^{k+\ell-j,*} & \xrightarrow{\hat{\Delta}} & \hat{E}^{k+\ell,*} & \xrightarrow{\hat{i}} & \hat{E}^{k,*} \times \hat{E}^{\ell,*} \\ \downarrow \hat{p} & & & & \\ \hat{E}^{k+\ell-2j,*} & & & & \end{array}$$

and

$$\begin{array}{ccccc} Z^{k+\ell-j} & \xrightarrow{\Delta_\infty} & Z^{k+\ell} & \xrightarrow{i_\infty} & Z^k \times Z^\ell \\ \downarrow p_\infty & & & & \\ Z^{k+\ell-2j} & & & & \end{array}$$

By [25, Proposition 3.5.4], for a smooth scheme X and a smooth closed subscheme Z of codimension c we have the following Gysin distinguished triangle

$$M_{gm}(X \setminus Z) \rightarrow M_{gm}(X) \rightarrow M_{gm}(Z)(c)[2c] \rightarrow M_{gm}(X \setminus Z)[1].$$

Put $m = k + \ell - 2j$. Then the diagram

$$\begin{array}{ccccccc}
M_{gm}(E^m) & \longrightarrow & M_{gm}(\hat{E}^{m,*}) & \longrightarrow & M_{gm}(Z^m)(1)[2] & \xrightarrow{+1} & \\
\uparrow p_* & & \uparrow \hat{p}_* & & \uparrow p_{\infty,*} & & \\
M_{gm}(E^{m+j}) & \longrightarrow & M_{gm}(\hat{E}^{m+j,*}) & \longrightarrow & M_{gm}(Z^{m+j})(1)[2] & \xrightarrow{+1} & \\
\uparrow \Delta_* & & \uparrow \hat{\Delta}_* & & \uparrow \Delta_{\infty,*} & & \\
M_{gm}(E^{m+2j})(-j)[-2j] & \longrightarrow & M_{gm}(\hat{E}^{m+2j,*})(-j)[-2j] & \longrightarrow & M_{gm}(Z^{m+2j})(-j+1)[-2j+2] & \xrightarrow{+1} & \\
\uparrow i'^* & & \uparrow \hat{i}'_* & & \uparrow i_{\infty,*} & & \\
M_{gm}(U^{k,\ell})(-j-1)[-2j-2] & \longrightarrow & M_{gm}(\hat{E}^{k,\ell,*})(-j-1)[-2j-2] & \longrightarrow & M_{gm}(Z^{k,\ell})(-j+1)[-2j+2] & \xrightarrow{+1} &
\end{array}$$

is commutative by [6, Proposition 4.10, Theorem 4.32]. Taking cohomology, we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
H_{\mathcal{M}}^{m+1}(\hat{E}^{m,*}, \mathbb{Q}(m+1)) & \longrightarrow & H_{\mathcal{M}}^{m+1}(E^m, \mathbb{Q}(m+1)) & \longrightarrow & H_{\mathcal{M}}^m(Z^m, \mathbb{Q}(m)) & & \\
\downarrow \hat{p}^* & & \downarrow p^* & & \downarrow p_{\infty}^* & & \\
H_{\mathcal{M}}^{m+1}(\hat{E}^{m+j,*}, \mathbb{Q}(m+1)) & \longrightarrow & H_{\mathcal{M}}^{m+1}(E^{m+j}, \mathbb{Q}(m+1)) & \longrightarrow & H_{\mathcal{M}}^m(Z^{m+j}, \mathbb{Q}(m)) & & \\
\downarrow \hat{\Delta}_* & & \downarrow \Delta_* & & \downarrow \Delta_{\infty,*} & & \\
H_{\mathcal{M}}^{k+\ell+1}(\hat{E}^{k,\ell,*}, \mathbb{Q}(m+j+1)) & \longrightarrow & H_{\mathcal{M}}^{k+\ell+1}(E^{k,\ell}, \mathbb{Q}(m+j+1)) & \longrightarrow & H_{\mathcal{M}}^{k+\ell}(Z^{k,\ell}, \mathbb{Q}(m+j)) & & \\
\downarrow \hat{i}'_* & & \downarrow i'^* & & \downarrow i_{\infty,*} & & \\
H_{\mathcal{M}}^{k+\ell+3}(\hat{E}^{k,\ell,*}, \mathbb{Q}(m+j+2)) & \longrightarrow & H_{\mathcal{M}}^{k+\ell+3}(U^{k,\ell}, \mathbb{Q}(m+j+2)) & \longrightarrow & H_{\mathcal{M}}^{k+\ell}(Z^{k,\ell}, \mathbb{Q}(m+j)) & &
\end{array}$$

Consider the subgroup $G' = \mu_2^k \rtimes \mathfrak{S}_k$ of G . Let ε'_k be the restriction of ε_k to G' , and let e'_k be the idempotent corresponding to ε'_k .

Denote $\tilde{\Xi}^{k,\ell,j}(\beta) = i'_* \circ \Delta_* \circ p^*(\text{Eis}^{k+\ell-2j}(\beta))$. Consider the image of $\tilde{\Xi}^{k,\ell,j}(\beta)$ under the residue map

$$\text{Res}^{k,\ell,j} : H_{\mathcal{M}}^{k+\ell+3}(U^{k,\ell}, \mathbb{Q}(k+\ell-j+2))^{(e'_k, e'_\ell)} \rightarrow H_{\mathcal{M}}^{k+\ell}(Z^{k,\ell}, \mathbb{Q}(k+\ell-j))^{(e'_k, e'_\ell)}$$

Note that $H_{\mathcal{M}}^{k+\ell}(Z^{k,\ell}, \mathbb{Q}(k+\ell))^{(e'_k, e'_\ell)}$ can be identified with $H_{\mathcal{M}}^k(Z^k, \mathbb{Q}(k))^{e'_k} \otimes_{\mathbb{Q}} H_{\mathcal{M}}^\ell(Z^\ell, \mathbb{Q}(\ell))^{e'_\ell} \simeq \mathcal{F}_N^k \otimes_{\mathbb{Q}} \mathcal{F}_N^\ell$.

Proposition 7.2. (1) If $j > 0$, then we have $\text{Res}^{k,\ell,j} \circ \tilde{\Xi}^{k,\ell,j} = 0$.

(2) If $j = 0$, then $\text{Res}^{k,\ell,0} \circ \tilde{\Xi}^{k,\ell,0}(\beta)$ is a nonzero multiple of $\omega_N^{k+\ell}(\beta) \otimes \omega_N^{k+\ell}(\beta)$.

Proof. (1) The image of Eisenstein symbol is contained in $H_{\mathcal{M}}^{k+\ell-2j}(Z^{k+\ell-j}, \mathbb{Q}(k+\ell-2j))^{e_{k+\ell-2j}}$. Let

$\Delta : \mathbb{G}_m^{k+\ell-j} \rightarrow \mathbb{G}_m^{k+\ell}$ be the diagonal embedding. We have

$$\Delta_* : H_{\mathcal{M}}^{k+\ell-2j}(\mathbb{G}_m^{k+\ell-j}, \mathbb{Q}(k+\ell-2j))^{e'_{k+\ell-2j}} \rightarrow H_{\mathcal{M}}^{k+\ell}(\mathbb{G}_m^{k+\ell}, \mathbb{Q}(k+\ell-j))^{e'_{k+\ell-2j}}.$$

By [21, 1.3.1 Lemma], one has

$$H_{\mathcal{M}}^{k+\ell}(\mathbb{G}_m^{k+\ell}, \mathbb{Q}(k+\ell-j))^{e'_{k+\ell-2j}} \simeq H_{\mathcal{M}}^{2j}(\mathbb{G}_m^{2j}, \mathbb{Q}(j)) \simeq \text{CH}^j(\mathbb{G}_m^{2j}).$$

Since $j > 0$, we have $\text{CH}^j(\mathbb{G}_m^{2j}) = 0$. Therefore $\Delta_* = 0$.

(2) This follows from the commutativity of the diagram. □

The closed embedding

$$i_{\text{cusp}} : Z^k \times E^\ell \hookrightarrow U^{k,\ell}$$

induces

$$H_{\mathcal{M}}^{k+\ell+1}(Z^k \times E^\ell, \mathbb{Q}(k+\ell+1)) \xrightarrow{i_{\text{cusp},*}} H_{\mathcal{M}}^{k+\ell+3}(U^{k,\ell}, \mathbb{Q}(k+\ell+2)).$$

Let us consider Gysin morphisms

$$\partial : M_{gm}(Z^\ell)(1)[2] \rightarrow M_{gm}(E^\ell)[1]$$

for the pair $(\hat{E}^{\ell,*}, Z^\ell)$ and

$$\partial' : M_{gm}(Z^k \times Z^\ell)(1)[2] \simeq M_{gm}(Z^k) \otimes M_{gm}(Z^\ell)(1)[2] \rightarrow M_{gm}(Z^k \times E^\ell)[1] = M_{gm}(Z^k) \otimes M_{gm}(E^\ell)[1]$$

for the pair $(Z^k \times \hat{E}^{\ell,*}, Z^k \times Z^\ell)$. By [6, Lemma 4.12], it follows that $\partial' = 1_{Z^k,*} \otimes \partial$. Therefore we have the following commutative diagram:

$$\begin{array}{ccc} H_{\mathcal{M}}^k(Z^k, \mathbb{Q}(k))^{e'_k} \otimes_{\mathbb{Q}} H_{\mathcal{M}}^{\ell+1}(E^\ell, \mathbb{Q}(\ell+1))^{e'_\ell} & \xrightarrow{1_{Z^k,*} \otimes \partial} & H_{\mathcal{M}}^k(Z^k, \mathbb{Q}(k))^{e'_k} \otimes_{\mathbb{Q}} H_{\mathcal{M}}^\ell(Z^\ell, \mathbb{Q}(\ell))^{e'_\ell} \\ \downarrow \mu & & \downarrow \simeq \\ H_{\mathcal{M}}^{k+\ell+1}(Z^k \times E^\ell, \mathbb{Q}(k+\ell+1))^{(e'_k, e'_\ell)} & \xrightarrow{\partial'} & H_{\mathcal{M}}^{k+\ell}(Z^{k,\ell}, \mathbb{Q}(k+\ell))^{(e'_k, e'_\ell)} \\ \downarrow i_{\text{cusp},*} & & \downarrow = \\ H_{\mathcal{M}}^{k+\ell+3}(U^{k,\ell}, \mathbb{Q}(k+\ell+2))^{(e'_k, e'_\ell)} & \xrightarrow{\text{Res}^{k,\ell,0}} & H_{\mathcal{M}}^{k+\ell}(Z^{k,\ell}, \mathbb{Q}(k+\ell))^{(e'_k, e'_\ell)}, \end{array}$$

where $\mu : H_{\mathcal{M}}^k(Z^k, \mathbb{Q}(k))^{e'_k} \otimes_{\mathbb{Q}} H_{\mathcal{M}}^{\ell+1}(E^\ell, \mathbb{Q}(\ell+1))^{e'_\ell} \rightarrow H_{\mathcal{M}}^{k+\ell+1}(Z^k \times E^\ell, \mathbb{Q}(k+\ell+1))^{(e'_k, e'_\ell)}$ is the exterior product.

Since ∂ is surjective, $1_{Z^k,*} \otimes \partial$ is also surjective. Hence $\text{Res}^{k,\ell,0} \circ i_{\text{cusp},*}$ is surjective. It follows that there exists an element $\xi_\beta \in H_{\mathcal{M}}^{k+\ell+1}(Z^k \times E^\ell, \mathbb{Q}(k+\ell+1))^{(e'_k, e'_\ell)}$ such that $\text{Res}^{k,\ell,0} \circ i_{\text{cusp},*}(\xi_\beta) = \text{Res}^{k,\ell,0}(\tilde{\Xi}^{k,\ell,0}(\beta))$. Now we define the generalized Beilinson-Flach element by

$$\text{BF}^{k,\ell,j}(\beta) := \begin{cases} \tilde{\Xi}^{k,\ell,j}(\beta) & \text{if } j > 0, \\ \tilde{\Xi}^{k,\ell,0}(\beta) - i_{\text{cusp},*}(\xi_\beta) & \text{if } j = 0. \end{cases}$$

From the definition, it is clear that

$$\text{BF}^{k,\ell,j}(\beta) \in \text{Im}[H_{\mathcal{M}}^{k+\ell+3}(\hat{E}^{k,\ell,*}, \mathbb{Q}(k+\ell-j+2))^{(e'_k, e'_\ell)} \rightarrow H_{\mathcal{M}}^{k+\ell+3}(U^{k,\ell}, \mathbb{Q}(k+\ell-j+2))^{(e'_k, e'_\ell)}].$$

Note that the map $H_{\mathcal{M}}^{k+\ell+3}(\hat{E}^{k,\ell,*}, \mathbb{Q}(k+\ell-j+2))^{(e'_k, e'_\ell)} \rightarrow H_{\mathcal{M}}^{k+\ell+3}(U^{k,\ell}, \mathbb{Q}(k+\ell-j+2))^{(e'_k, e'_\ell)}$ is injective.

8. EXTENSIONS TO THE BOUNDARY

To extend the motivic element $\Xi^{k,\ell,j}(\beta)$ to the boundary of the Kuga-Sato varieties, we use the following proposition.

Proposition 8.1. *We have an isomorphism*

$$H_{\mathcal{M}}^{k+\ell+3}(\hat{E}^{k,*} \times \hat{E}^{\ell,*}, \mathbb{Q}(k+\ell-j+2))^{(e'_k, e'_\ell)} \simeq H_{\mathcal{M}}^{k+\ell+3}(\overline{\overline{E}}^k \times \overline{\overline{E}}^\ell, \mathbb{Q}(k+\ell-j+2))^{(e_k, e_\ell)}.$$

To show the proposition, we prepare the following lemma.

Lemma 8.2. $M_{gm}(\hat{E}^{k,*})^{e'_k} \simeq M_{gm}(\overline{\overline{E}}^k)^{e_k}$ in $DM_{gm}^{\text{eff}}(\mathbb{Q})_{\mathbb{Q}}$.

Proof of Lemma 8.2. Let $\overline{\overline{E}}^{k,\infty}$ be the complement of the smooth scheme E^k in the smooth proper scheme $\overline{\overline{E}}^k$. Let $\overline{\overline{E}}^{k,\infty,\text{reg}}$ be the intersection of $\overline{\overline{E}}^{k,\infty}$ with the non-singular part $\overline{E}^{k,\text{reg}}$ of \overline{E}^k and $\overline{\overline{E}}^{k,\infty,0} \subset \overline{\overline{E}}^{k,\infty,\text{reg}}$ the intersection of $\overline{\overline{E}}^{k,\infty,\text{reg}}$ with $\hat{E}^{k,*}$. Note that the morphism $\overline{\overline{E}}^k \rightarrow \overline{E}^k$ is an isomorphism over $\overline{E}^{k,\text{reg}}$ by [21, Theorem 3.1.0 (ii)], hence $\overline{E}^{k,\text{reg}}$ can be identified with a subscheme of $\overline{\overline{E}}^k$ and the open immersion $\overline{E}^{k,\text{reg}} \hookrightarrow \overline{\overline{E}}^k$ induces an isomorphism

$$M_{gm}^c(\overline{\overline{E}}^k)^{e_k} \xrightarrow{\sim} M_{gm}^c(\overline{E}^{k,\text{reg}})^{e_k}$$

by [26, Remark 3.8 (a)]. Also the connected component $\hat{E}^{k,*}$ is identified with a subscheme of $\overline{E}^{k,\text{reg}}$ by [21, Theorem 3.1.0 (iii)]. From these facts and [26, Proof of Theorem 3.3], one has

$$M_{gm}^c(\overline{\overline{E}}^k)^{e_k} \xrightarrow{\sim} M_{gm}^c(\overline{\overline{E}}^{k,\infty,\text{reg}})^{e_k} \xrightarrow{\sim} M_{gm}^c(\overline{\overline{E}}^{k,\infty,0})^{e'_k}.$$

By [25, Proposition 4.1.5] we have the distinguished triangles:

$$\begin{array}{ccccccc}
M_{gm}^c(\overline{\overline{E}}^k)^{e_k} & \longrightarrow & M_{gm}^c(E^k)^{e_k} & \longrightarrow & M_{gm}^c(\overline{\overline{E}}^{k,\infty})^{e_k}[1] & \xrightarrow{+1} & \\
\downarrow \simeq & & \downarrow = & & \downarrow \simeq & & \\
M_{gm}^c(\overline{E}^{k,\text{reg}})^{e_k} & \longrightarrow & M_{gm}^c(E^k)^{e_k} & \longrightarrow & M_{gm}^c(\overline{\overline{E}}^{k,\infty,\text{reg}})^{e_k}[1] & \xrightarrow{+1} & \\
\downarrow & & \downarrow & & \downarrow \simeq & & \\
M_{gm}^c(\hat{E}^{k,*})^{e'_k} & \longrightarrow & M_{gm}^c(E^k)^{e'_k} & \longrightarrow & M_{gm}^c(\overline{\overline{E}}^{k,\infty,0})^{e'_k}[1] & \xrightarrow{+1} &
\end{array}$$

Moreover one has

$$M_{gm}^c(E^k)^{e_k} \xrightarrow{\simeq} M_{gm}^c(E^k)^{e'_k},$$

since we have a decomposition $E^k = \coprod_{0 \leq q \leq k} \mathring{Y}_q^k$ of E^k into locally closed subsets which are invariant under the action of $\mathfrak{S}_{k+1} \cdot (\mathbb{Z}/N\mathbb{Z})^{2k}$ as in [20, Proof of 4.2 Theorem], where

$$\mathring{Y}_q^k = \{(x_1, \dots, x_k) \in E^k \mid \text{exactly } q \text{ of the } x_i\text{'s are in } E[N]\}.$$

From this fact, it follows that the inclusion $\hat{E}^{k,*} \hookrightarrow \overline{E}^{k,\text{reg}}$ induces

$$M_{gm}^c(\overline{E}^{k,\text{reg}})^{e_k} \xrightarrow{\simeq} M_{gm}^c(\hat{E}^{k,*})^{e'_k}$$

and hence

$$M_{gm}^c(\overline{\overline{E}}^k)^{e_k} \xrightarrow{\simeq} M_{gm}^c(\hat{E}^{k,*})^{e'_k}.$$

By duality for smooth schemes [25, Theorem 4.3.7 3], we have

$$M_{gm}(\overline{\overline{E}}^k)^{e_k} \xleftarrow{\simeq} M_{gm}(\overline{E}^{k,\text{reg}})^{e_k} \xleftarrow{\simeq} M_{gm}(\hat{E}^{k,*})^{e'_k}.$$

□

Proof of Proposition 8.1. Applying Künneth formula [25, Proposition 4.1.7], we have

$$M_{gm}(\hat{E}^{k,*} \times \hat{E}^{\ell,*})^{(e'_k, e'_\ell)} \simeq M_{gm}(\overline{\overline{E}}^k \times \overline{\overline{E}}^\ell)^{(e_k, e_\ell)}.$$

By Voevodsky's definition of motivic cohomology, we have

$$H_{\mathcal{M}}^i(\hat{E}^{k,*} \times \hat{E}^{\ell,*}, \mathbb{Q}(j))^{(e'_k, e'_\ell)} \simeq H_{\mathcal{M}}^i(\overline{\overline{E}}^k \times \overline{\overline{E}}^\ell, \mathbb{Q}(j))^{(e_k, e_\ell)}$$

for any i, j . This completes the proof. □

Via the isomorphism in Proposition 8.1, we can consider $\text{BF}^{k,\ell,j}(\beta)$ as an element of $H_{\mathcal{M}}^{k+\ell+3}(\overline{\overline{E}}^k \times \overline{\overline{E}}^\ell, \mathbb{Q}(k+\ell-j+2))^{(e_k, e_\ell)}$.

Proposition 8.3. *We have $\langle r_{\mathcal{D}}(\text{BF}^{k,\ell,j}(\beta)), \Omega_{f,g} \rangle = \langle r_{\mathcal{D}}(\Xi^{k,\ell,j}(\beta)), \Omega_{f,g} \rangle$.*

Proof. The regulator map is compatible with the contravariance of morphisms by [16, 8.1.a)]. Therefore it is enough to show that $\langle r_{\mathcal{D}}(i_{\text{cusp},*}(\xi_\beta)), \Omega_{f,g} \rangle = 0$.

By Jannsen's formula for the regulator in [14, page 45], the image of the regulator map is represented by an integral along $Z^k \times E^\ell$. Since f^* is a cusp form, the differential form ω_{f^*} vanishes on the cycle Z^k . Hence the differential form $\Omega_{f,g}$ vanishes on the cycle $Z^k \times E^\ell$. Therefore the regulator integral vanishes. □

9. APPLICATION TO BEILINSON'S CONJECTURE

Consider the projection to the $f \otimes g$ -component

$$\text{pr}_{f,g} : H_{\mathcal{D}}^{k+\ell+3}(\overline{\overline{E}}_{\mathbb{R}}^k \times \overline{\overline{E}}_{\mathbb{R}}^\ell, \mathbb{Q}(k+\ell+2-j))^{(e_k, e_\ell)} \rightarrow H_{\mathcal{D}}^{k+\ell+3}(M(f \otimes g), \mathbb{R}(k+\ell+2-j)).$$

Our results admit the following consequence for Beilinson's conjecture for the motive $M(f \otimes g)(k+\ell+2-j)$.

Theorem 9.1. *Let $f \in S_{k+2}(\Gamma_1(N_f), \chi_f)$ and $g \in S_{\ell+2}(\Gamma_1(N_g), \chi_g)$ be newforms with $k, \ell \geq 0$. Let N be an integer divisible by N_f and N_g , and let j be an integer satisfying $0 \leq j \leq \min(k, \ell)$. Assume $\chi_f \chi_g \neq 1$ and $R_{f,g,N}(j+1) \neq 0$. Then there is an element $\alpha \in H_{\mathcal{M}}^{k+\ell+3}(\overline{E}^k \times \overline{E}^\ell, \mathbb{Q}(k+\ell+2-j))^{(e_k, e_\ell)}$ such that*

$$\mathrm{pr}_{f,g} \circ r_{\mathcal{D}}(\alpha) = L^*(M(f \otimes g)(k+\ell+2-j)^\vee(1), 0) \cdot t \pmod{K_{f,g}^\times},$$

where t is a generator of the $K_{f,g}$ -rational structure in $H_{\mathcal{D}}^{k+\ell+3}(M(f \otimes g), \mathbb{R}(k+\ell+2-j))$.

Proof. Note that $R_{f,g,N}(j+1) \neq 0$ is equivalent to $R_{f^*,g^*,N}(j+1) \neq 0$. The theorem follows from Lemma 2.2, Proposition 6.4, Proposition 8.3 and the fact that $\Omega_{f,g}$ is a $K_{f,g}^\times$ -rational multiple of Ω . \square

Using the compatibility of Beilinson's conjecture with respect to the functional equation [19, (2.2.2)], we get the following corollary.

Corollary 9.2. *Under the assumptions of Theorem 9.1, the weak version of Beilinson's conjecture for $L(f \otimes g, k+\ell+2-j)$ holds.*

Remark 9.3. (1) The factor $R_{f,g,N}(j+1)$ is a product of local terms $R_{f,g,p}(j+1)$, where p runs through the prime factors of N . If p divides exactly one of the integers N_f and N_g , then $R_{f,g,p}(s) = 1$ by [13, Theorem 15.1]. If p divides N but doesn't divide $N_f N_g$, then $R_{f,g,p}(s) = 1 - \chi_f(p)\chi_g(p)p^{k+\ell+2-2s}$ by [23, Lemma 1] and it may happen that $R_{f,g,p}(j+1) = 0$, for example if $j = k = \ell$ and $p = 1 \pmod{\mathrm{lcm}(N_f, N_g)}$. Therefore, it is best to choose $N = \mathrm{lcm}(N_f, N_g)$ in Theorem 9.1. Moreover, it is easy to see that $R_{f,g,N}(j+1) \neq 0$ if $k+\ell-2j \notin \{0, 1, 2\}$ by [13, Theorem 15.1].

(2) The assumption $R_{f,g,N}(j+1) \neq 0$ is necessary. We give an example. There is a newform f of weight 8, level 39 with character $(\frac{13}{\cdot})$ such that $a_3(f) = -27$ (it is called 39.8.5a in the modular forms database <http://www.lmfdb.org/>). Also there is a newform g of weight 8, level 3 with trivial character such that $a_3(g) = -27$ (it is called 3.8.a in the modular forms database). Let $\pi_f = \bigotimes'_v \pi_{f,v}$ and $\pi_g = \bigotimes'_v \pi_{g,v}$ be the automorphic representations generated by f and g . Then it is easy to see that $\pi_{f,3}$ and $\pi_{g,3}$ are special representations of the form $\mathrm{sp}(\sigma_{f,3} |^{-\frac{1}{2}}, \sigma_{f,3} |^{\frac{1}{2}})$ and $\mathrm{sp}(\sigma_{g,3} |^{-\frac{1}{2}}, \sigma_{g,3} |^{\frac{1}{2}})$, where $\sigma_{f,3}$ and $\sigma_{g,3}$ are unramified characters of \mathbb{Q}_3^\times satisfying $\sigma_{f,3}(3) = \sigma_{g,3}(3) = -1$. By [13, Theorem 15.1], we have

$$L(\pi_{f,3} \otimes \pi_{g,3}, s) = L(\sigma_{f,3}\sigma_{g,3}, s)L(\sigma_{f,3}\sigma_{g,3}, s+1) = (1-3^{-s})(1-3^{-s-1}).$$

Hence the Euler factor of $L(f \otimes g, s) = L(\pi_f \otimes \pi_g, s-7)$ at 3 is given by $(1-a_3(f)a_3(g)3^{-s+1})(1-a_3(f)a_3(g)3^{-s}) = (1-3^6 \cdot 3^{-s+1})(1-3^6 \cdot 3^{-s})$. On the other hand, the Euler factor of $D(f, g, s)$ at 3 is $1-a_3(f)a_3(g)3^{-s} = 1-3^6 \cdot 3^{-s}$. Therefore $R_{f,g,N}(s) = 1-3^6 \cdot 3^{-s+1} = 1-3^7 \cdot 3^{-s}$. If $j = 6$, then $R_{f,g,39}(j+1) = 0$.

REFERENCES

- [1] S. Baba and R. Sreekantan, *An analogue of circular units for products of elliptic curves*, Proc. Edinb. Math. Soc. (2) **47** (2004), no. 1, 35–51.
- [2] A. A. Beilinson, *Higher regulators and values of L-functions*, J. Soviet Math. **30** (1985), 2036–2070.
- [3] A. A. Beilinson, *Higher regulators of modular curves*, Applications of algebraic K-theory to algebraic geometry and number theory, Part I, Proceedings of Summer Research Conference held June 12–18, 1983, in Boulder, Colorado, Contemporary Mathematics 55, American Mathematical Society, Providence, Rhode Island, 1–34.
- [4] M. Bertolini, H. Darmon and V. Rotger, *Beilinson-Flach elements and Euler systems I: syntomic regulators and p-adic Rankin L-series*, Journal of Algebraic Geometry **24** (2015), 355–378.
- [5] M. Bertolini, H. Darmon and V. Rotger, *Beilinson-Flach elements and Euler systems II: the Birch and Swinnerton-Dyer conjecture for Hasse-Weil-Artin L-series*, Journal of Algebraic Geometry, to appear.
- [6] F. Déglise, *Around the Gysin triangle II*, Doc. Math. **13** (2008), 613–675.
- [7] P. Deligne, *Valeurs de fonctions L et périodes d'intégrales*, Proc. Sympos. Pure Math., vol. 33. Automorphic Forms. Representations and L-functions (Oregon State Univ. Corvallis. 1977), Part 2, Amer. Math. Soc. Providence. R.I., 1979, pp. 313–346.
- [8] C. Deninger, *Higher regulators and Hecke L-series of imaginary quadratic fields I*, Invent. Math. **96** (1989), no. 1, 1–69.
- [9] C. Deninger, *Extensions of motives associated to symmetric powers of elliptic curves and to Hecke characters of imaginary quadratic fields*, in: F. Catanese (ed.): Proc. Arithmetic Geometry, Cortona, 1994, Cambridge University Press, 1997, pp. 99–137.
- [10] C. Deninger and A. Scholl, *The Beilinson conjectures*, In: L-functions in Arithmetic, ed. Coates-Taylor (Cambridge, 1991), 173–209.

- [11] M. Flach, *A finiteness theorem for the symmetric square of an elliptic curve*, Invent. Math. **109** (1992), no. 2, 307–327.
- [12] A. Huber and G. Kings, *Dirichlet motives via modular curves*, Ann. Sci. Éc. Norm. Sup. **32** (1999), 313–345.
- [13] H. Jacquet, *Automorphic forms on $GL(2)$. part II*, Lecture notes in Mathematics, Vol. 278, Springer-Verlag, Berlin, 1972.
- [14] U. Jannsen, *Deligne homology, Hodge-D-conjecture, and motives*, in Beilinson’s Conjectures on Special Values of L -functions (Academic Press, Boston, MA, 1988), 305–372.
- [15] U. Jannsen, *Continuous étale cohomology*, Math. Ann. **280** (1988), 207–245.
- [16] U. Jannsen, *Mixed motives and algebraic K-theory*, Springer Lect. Notes in Math. **1400** (1990), xiii+246 pp.
- [17] G. Kings, D. Loeffler and S.L. Zerbes, *Rankin-Eisenstein classes for modular forms*, preprint.
- [18] A. Lei, D. Loeffler and S. L. Zerbes, *Euler systems for Rankin-Selberg convolutions of modular forms*, Ann. of Math. (2) **180** (2014), no. 2, 653–771.
- [19] J. Nekovář, *Beilinson’s conjectures*, In: Motives (Seattle, WA, 1991), Proc. Symp. Pure Math., **55** Part I, 537–570, Amer. Math. Soc., Providence, RI, 1994.
- [20] N. Schappacher and A. J. Scholl, *The boundary of the Eisenstein symbol*, Math. Ann. **290** (1991), 303–321.
- [21] A. J. Scholl, *Motives for modular forms*, Invent. Math. **100** (1990), 419–430.
- [22] A. J. Scholl, *Integral elements in K-theory and products of modular curves*, The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), 467–489, NATO Sci. Ser. C Math. Phys. Sci., 548, Kluwer Acad. Publ., Dordrecht, 2000.
- [23] G. Shimura, *The special values of the zeta functions associated with cusp forms*, Comm. Pure Appl. Math. **29** (1976), no. 6, 783–804.
- [24] G. Shimura, *On the periods of modular forms*, Math. Ann. **229** (1977), no. 3, 211–221.
- [25] V. Voevodsky, *Triangulated categories of motives*, in Cycles, transfers, and motivic homology theories, Annals of Mathematics Studies, vol. 143 (Princeton University Press, Princeton, NJ, 2000).
- [26] J. Wildeshaus, *Chow motives without projectivity*, Compositio Math. **145** (2009), 1196–1226.

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